MODULE 1

Random experiment

* Random experiment is an experiment whose all possible outcomes are known and which can be repeated under identical conditions.
* In random experiment, we cannot predict the outcome in advance.

eg: Tossing a coin, Tossing a die

sample space

* sample space of a random experiment is the set of all possible outcomes of the experiment. It is denoted by 'S'.
* The elements of S are called sample point.

eg: If an unbiased coin is tossed, then the sample space, $S = \{H, T\}$.

If you are tossing a die, then the sample space, $S = \{1, 2, 3, 4, 5, 6\}$.

EVENT

Any subset of the sample space of a random experiment is called event.

Trial

Any particular performance of a random expt is called a trial.

Random variable

* A function whose domain is the set of possible outcomes, and whose range is a subset of the set of reals. such a function is called a random variable.

* A random variable is a function that assign a real number $x(s)$ to every $s \in S$, where S is the sample space.

* The set of all possible real values $\{x(s); s \in S\}$ is called the Range space and is denoted by R_x.

* Random variable mainly divided in to 2 types.
Discrete random variable
Continuous random variable.

Discrete Random Variable

A random variable is said to be discrete if it's range is finite or countably infinite.

eg: consider a random expt of tossing a coin until a head appear, then the sample space;

$S = \{H, TH, TTH, \cdots\}$

Let X represents the no. of toss required to get a head. ∴ X takes values 1, 2, 3, ... (range space) it is ~~a~~ infinite but countable. here the random variable X is discrete.

Probability distribution of X ($p(x_i)$ or p_i)

x	$p(x)$
2	1/36
3	2/36
4	3/36
5	4/36
6	5/36
7	6/36
8	5/36
9	4/36
10	3/36
11	2/36
12	1/36

$\Sigma p(x) = 1$

The probability associated with all the values of random variable X ~~are~~ shown in the above table. From the above table we can assign probabilities to each of the random variable X such that the total probability = 1.

The above distribution is called probability distribution of random variable X.

Probability Mass function (PMf)

If X is a discrete random variable, the function given by, $P(X=x_i)=p_i$, for each x_i within the range of X is called the probability mass function of the random variable X, provided p_i satisfy the following condition,

1) $p_i \geq 0$ for all i
2) $\sum_{\text{all } i} p_i = 1$

* Domain of probability function is $(-\infty, \infty)$ and range is $[0, 1]$.

cumilative distribution functions

the distribution function of a random variable X is defined by the formula

$F(x) = P(X \leq x)$ is called cumilative distribution function of X

* consider the random expt of tossing 3 coins,

$S = \{ HHH, HHT, HTH, HTT, THH, THT, TTH, TTT \}$

Let X represents the no. of heads, then X takes values 0, 1, 2, 3. Then the probability distribution of X is given by,

X	0	1	2	3
$p(x_i)$	1/8	3/8	3/8	1/8

$\sum_{x} p(x_i) = 1$

Distribution function of random variable X is given by, $F(x) = P(X \leq x)$

when $x = 0$,

$F(0) = P(X \leq 0) = P(X = 0) = 1/8$

when $x = 1$,

$F(1) = P(X \leq 1) = P(X = 0) + P(X = 1)$
$= 4/8$

when $x = 2$,

$F(2) = P(X \leq 2) = 1/8 + 3/8 + 3/8 = 7/8$

when $x = 3$,

$F(3) = P(X \leq 3) = 1/8 + 3/8 + 3/8 + 1/8 = 1$

Defn:

If X is a discrete random variable, the the function given by $F(x) = P(X \leq x) = \sum_{t \leq x} P(t)$

$, -\infty \leq x \leq \infty$

$p(t) = p(X = t)$

is called the distribution function of X

when $x = 0$,

$$F(0) = \sum_{t \leq 0} p(0) = p(0) = 1/8$$

when $x = 1$,

$$F(1) = \sum_{t \leq 1} p(1) = p(1) + p(0)$$

$$= 1/8 + 3/8 = 4/8$$

when $x = 2$,

$$F(2) = \sum_{t \leq 2} p(2) = p(0) + p(1) + p(2)$$

$$= 7/8$$

when $x = 3$,

$$F(3) = \underline{1}$$

Mathematical expectation of X

If X is a discrete random variable with probability mass function $p(X = x_i) = P_i$, $i = 1, 2, \ldots$ then the mathematical expectation of X or the arithmetic mean of X, denoted as $E(X)$, is defined as,

$$E(X) = \sum_i x_i p_i$$

$$E(X^2) = \sum_i x_i^2 p_i$$

Mean (μ) = $\sum_{i=1}^{n} x_i p_i$

Variance (σ^2) = $\sum_{i=1}^{n} (x_i - \mu)^2 p_i$

Binomial distribution

If a discrete random variable X can take the values 0, 1, 2, ... n such that $P(X=x)$ is given by,

$$B(x : n, p) = {}^{n}C_{x}\, p^{x} q^{n-x}, \quad x = 0, 1, 2, \ldots n.$$

where $p+q=1$,

Here n and p are called parameters of the binomial distribution.

This is called Binomial distribution bcoz the probabilities as the random variable X takes values 0, 1, 2, ... n are q^n, ${}^{n}C_1 q^{n-1} p$, ${}^{n}C_2 q^{n-2} p^2$, ... p^n which are the successive terms of the binomial expansion $(q+p)^n$.

? An agricultural cooperative claims that 90% of the watermelons shipped out are ripe and ready to eat. Find the probabilities that among 18 watermelons shipped out

(a) All 18 are ripe and ready to eat

(b) Atleast 16 are ripe and ready to eat

(c) Atmost 14 are ripe and ready to eat.

A). $n = 18$

$p = 90\% = 0.9$

a) $x = 18$

$B(18: 18, 0.9) = {}^{18}C_{18} (0.9)^{18} (1-0.9)^{0}$

$= 0.15$

b) $n = 18$

$p = 0.9$

$x = 16, 17, 18$

$B(x: n, p) = {}^{n}C_{x}\, p^{x} q^{n-x}$

$B(16: 18, 0.9) = {}^{18}C_{16}\, 0.9^{16} (1-0.9)^{2}$

$= 0.2835.$

$B(17: 18, 0.9) = {}^{18}C_{17}\, 0.9^{17} (1-0.9)^{1}$

$= 0.3002.$

$B(18: 18, 0.9) = 0.15$

$P(X \geq 16) = 0.2835 + 0.3002 + 0.15$

$= \underline{0.7338}$

c). $X = 1, 2, \cdots 14$

we've to find $P(X \leq 14)$

$P(X \leq 14) = 1 - P(X \geq 15)$

$B(15 : 18, 0.9) = {}^{18}C_{15}\, 0.9^{15} (1 - 0.9)^3$

$= 0.168$

$B(16 : 18, 0.9) = 0.2835$

$B(17 : 18, 0.9) = 0.3002$

$B(18 : 18, 0.9) = 0.15$

Total probability $= 0.9017$

$P(X \leq 14) = \underline{0.0983}$

2. If the probability is 0.40 that steam will condense in a thin-walled aluminium tube at 10 atm pressure, use the formula for the binomial distribution to find the probability that, under the stated conditions, steam will condense in.

1. 4 of 12 such tubes
2. Atmost 2 of 10 such tubes.

A) $p_1 = 0.4$

a) $n = 12$

$x = 4$

$B(x: n, p) = {}^{n}C_{x}\, p^{x} q^{n-x}$

$= {}^{12}C_{4}\, 0.4^{4}\, (0.6)^{8}$

$= 0.2128$

$P(X=4) = 0.2128$

b) $P(X \leq 2) = ?$

$n = 10$

$B(0: 10, 0.4) = {}^{10}C_{0}\, (0.4)^{0}\, (0.6)^{10}$

$= 0.0064.$

$B(1: 10, 0.4) = {}^{10}C_{1}\, (0.4)^{1}\, (0.6)^{9}$

$= 0.0403$

$B(2: 10, 0.4) = {}^{10}C_{2}\, (0.4)^{2}\, (0.6)^{8}$

$= 0.1209$

$P(X \leq 2) = 0.1676$

2 Human error is given as the reason for 75% of all accidents in a plant. use the formula for the binomial distribution to find the probability that human error will be given as the reason for 2 of the next four accidents.

A $n = 4$

$x = 2.$

$p = 0.75.$

$B(2: 4, 0.75) = {}^4C_2 (0.75)^2 (1-0.75)^2$

$= \underline{\underline{0.2109}}$

2 A manufacturer of metal pistons finds that on the avg 12% of his pistons are rejected bcoz they are either oversize or undersize. what is the probability that a batch of 10 pistons will contain.

a) no more than 2 rejects

b) at least 2 rejects.

A) $n = 10$

$p = 0.12$

a) $P(X \leq 2) = ?$

$B(0; 10, 0.12) = {}^{10}C_0 (0.12)^0 (0.88)^{10}$

$= 0.2785$

$B(1: 10, 0.12) = {}^{10}C_1 (0.12)^1 (0.88)^9$

$= 0.3797$

$B(2: 10, 0.12) = {}^{10}C_2 (0.12)^2 (0.88)^8$

$= 0.233$

$P(X \leq 2) = 0.8912$

b) $P(X \geq 2) = ?$

$P(X \geq 2) = 1 - P(X < 2)$

$= 1 - P(X=1) + P(X=0)$

$= 1 - 0.6582$

$= 0.3418$

2. Ten fair coins are thrown simultaneously. find probability of getting atleast 7 heads.

A). $P(X \geq 7) = ?$

$n = 10$

$P = 0.5$

~~$P(X=7)$~~

$B(7: 10, 0.5) = {}^{10}C_7 (0.5)^7 (0.5)^3 = 0.1172$

$B(8: 10, 0.5) = {}^{10}C_8 (0.5)^{10}$

$= 0.0439$

$B(9: 10, 0.5) = {}^{10}C_9 (0.5)^{10}$

$= 0.0098$

$B(10: 10, 0.5 = {}^{10}C_{10} (0.5)^{10}$

$= 0.00097$

$P(X \geq 7) = \underline{\underline{0.17177}}$

Note:

* Let X be a discrete random variable with the set of possible values $x_1, x_2, \dots, x_n$ and $g(X)$ be a function of X. Then $E[g(x)] = g(x_i) p(x_i)$

* $E(aX+b) = aE(X) + E(b)$

$= aE(X) + b$, where a and b are constants and X a random variable.

* Let X be a random variable and a and b be constants. Then $var(aX+b) = a^2 Var(X)$

* variance, $\sigma^2 = \sum_{i=1}^{n} (x_i - \mu)^2 p(x_i)$

$= E[(X-\mu)^2]$

$= E(X^2) - 2E(X\mu) + \mu^2$

$= E(X^2) - 2\mu E(X) + \mu^2 \quad \because \mu = E[X]$

$= E(X^2) - 2[E(X)]^2 + [E(X)]^2$

$= E(x^2) - [E(x)]^2$

So variance, $Var(X) = E(x^2) - [E(x)]^2$

* $E(\text{constant}) = \text{constant}.$

2 Find the mean and variance of the following data

x	1	2	3	4	5	6	7
$p(x)$	0.01	0.03	0.13	0.25	0.39	0.17	0.02.

Also calculate the mean and variance for the random variable $Y = 2x + 3$

A).

x	$p(x)$	$x_i p_i$	x^2.	$x^2 p_i$
1	0.01	0.01	1	0.01
2	0.03	0.06	4	0.12
3	0.13	0.39	9	1.17
4	0.25	1	16	4
5	0.39	1.95	25	9.75
6	0.17	1.02	36	6.12.
7	0.02	0.14	49	0.98
		4.57		

$Mean = E(x) = \Sigma x_i p_i = 4.57$

$$variance\ (\sigma^2) = E(x^2) - E(x)^2$$

$$E(x^2) = \sum x^2 p(x_i).$$

$$= 22.15$$

$$var(x) = 22.15 - (4.57)^2$$

$$= 1.265$$

b) $Y = 2X + 3$

$$E(Y) = 2E(X) + 3$$

$$= 2 \times 4.57 + 3$$

$$= 12.14$$

$$var(Y) = 2^2 var(X)$$

$$= 4 \times 1.265$$

$$= 5.06$$

Bernoulli experiment

A random experiment consisting of only two possible outcomes, namely success and failure (a random variable x is assigned with 1 if the outcome is a success and 0 if the outcome is a failure) is called a Bernoulli experiment.

eg:

Tossing a coin & expecting a head

Rolling a die & observing whether a 6 occurs.

2. 60% of people who purchase sports cars are men. if 10 sports car owners are randomly selected, find the probability that exactly 7 are men.

A). $n = 10$

$p = 60/100 = 0.6$

we've to find P(exactly 7 are men)

$P(X=7) = ?$

$B(x: n, p) = {}^{n}C_{x}\, p^{x}\, q^{n-x}$

$B(7: 10, 0.6) = {}^{10}C_{7}\,(0.6)^{7}\,(0.4)^{3}$

$= 0.2149$

3. It has been claimed that in 60% of all solar-heat installations the utility bill is reduced by atleast one-third. Accordingly, what are the probabilities that the utility bill will be reduced by atleast one-third in

(a) four of five installations.

b) atleast four of five constallation.

A). $p = 0.6$

$n = 5$

$x = 4$

$$B(4:5,0.6) = {}^{n}C_{x}\, p^{x} q^{n-x}$$

$$= {}^{5}C_{4}\, 0.6^{4} (0.4)^{1}$$

$$= \underline{\underline{0.2592}}$$

b) $P(X \geq 4) = ?$

$n = 5$

$x = 4, 5$

$$B(x:n,p) = {}^{n}C_{x}\, p^{x} q^{n-x}$$

$$B(5:5,0.6) = {}^{5}C_{5} (0.6)^{5} (0.4)^{0}$$

$$= 0.0777$$

$$P(X \geq 4) = 0.0777 + 0.2592$$

$$= \underline{\underline{0.3369}}$$

Mean & Variance of binomial distribution

<u>Mean:</u>

Let $X=x$ be the random variable with the values $x = 0, 1, 2, \dots, n$. Then $p(x) = \binom{n}{x} p^{x} (1-p)^{n-x}$

we know that Mean $\mu = \sum_{x=0}^{n} x f(x)$

$$= \sum_{x=0}^{n} x \, {}^{n}C_{x} \, p^{x} (1-p)^{n-x}$$

$$= \sum_{x=1}^{n} x \, {}^{n}C_{x} \, p^{x} (1-p)^{n-x} \quad \left[\text{for } x=0 \quad \sum x f(x) = 0\right]$$

$$= \sum_{x=1}^{n} x \frac{n!}{x!\,(n-x)!} p^{x} (1-p)^{n-x}$$

$$= \sum_{x=1}^{n} x \frac{n(n-1)!}{x(x-1)!\,((n-1)-(x-1))!} p \cdot p^{x-1} (1-p)^{(n-1)-(x-1)}$$

$$= np \sum_{x=1}^{n} \frac{(n-1)!}{(x-1)!\,((n-1)-(x-1))!} p^{x-1} (1-p)^{((n-1))-(x-1)}$$

$$= np \sum_{x=1}^{n} {}^{(n-1)}C_{(x-1)} \, p^{x-1} (1-p)^{(n-1)-(x-1)}$$

$=$ [sub. $M = n-1$

$y = x-1$ (~~x=~~y

when $x = 1$, $y = 0$

when $x = n$, $y = n-1 = M$]

$$E(X) = np \sum_{y=0}^{n-1} {}^{M}C_{y} \, p^{y} (1-p)^{m-y}$$

The summation can be recognized as the sum of all the terms of a binomial distribution with the parameters M and P.

Hence the summation equal to 1,

$\therefore \quad E(x) = np * 1$

$\qquad\qquad = \underline{\underline{np}}$

Mean = np

<u>Variance</u>

Let $X = x$ be the r.v with the values, $x = 0, 1, 2, \dots, n$ Then $p(x) = {}^{n}C_{x}\, p^{x}(1-p)^{n-x}$

we've

$$\sigma^2 = E(x^2) - (E(x))^2$$

To determine $E(x^2)$

consider $E(X(X-1)) = \sum_{x=0}^{n} x(x-1) f(x)$

$$= \sum_{x=2}^{n} x(x-1)\, {}^{n}C_{x}\, p^{x}(1-p)^{n-x}$$

$$= \sum_{x=2}^{n} x(x-1) \frac{n(n-1)(n-2)!}{x(x-1)(x-2)!\,((n-2)-(x-2))!} p^{2} p^{x-2} (1-p)^{n-2-(x-2)}$$

$$= n(n-1)p^2 \sum_{x=2}^{n} {}^{(n-2)}C_{(x-2)}\, p^{x-2} (1-p)^{n-2-(x-2)}$$

[sub, $M = n-2$

$y = x-2$.

when $x = 2$, $y = 0$

when $x = n$, $y = M$]

$$= n(n-1)p^2 \sum_{y=0}^{m} {}^{m}C_{y}\, p^y (1-p)^{m-y}$$

$$E(X(X-1)) = n(n-1)p^2$$

$$E(X(X-1)) = E(X^2) - E(X) = n^2p^2 - np^2$$

$$E(X^2) = n^2p^2 - np^2 + E(X)$$

$$= n^2p^2 - np^2 + np \qquad [\because E(X) = np]$$

$$Var(X) = E(X^2) - (E(X))^2$$

$$= n^2p^2 - np^2 + np - n^2p^2$$

$$= np - np^2$$

$$= np(1-p)$$

Variance $= np(1-p)$

2. The mean and variance of a binomial variable X are 16 and 8 respectively. Find $P(X=0)$ and $P(X=1)$

A). Given that,

mean of binomial distribution $(E(X)) = np = 16$

$np = 16$

variance, $np(1-p) = 8$

$16(1-p) = 8$

$1-p = \frac{1}{2}$

$p = \frac{1}{2}$

$n = 16/p = 32$

$n = 32$ & $p = \frac{1}{2}$

$P(X=0) = {}^{n}C_{x}\, p^{x} (1-p)^{n-x}$

$= {}^{32}C_{0}\, p^{0} (1-p)^{32}$

$= 1 \times (\frac{1}{2})^{32}$

$= \left(\frac{1}{2}\right)^{32}$

$P(X=1) = {}^{32}C_{1} (\frac{1}{2})^{1} (\frac{1}{2})^{31}$

$= 32 \cdot (\frac{1}{2})^{32}$

2. In a binomial distribution consisting of 6 independent trials, probabilities of 1 & 2 successes are 0.28336 and 0.0506 respectively. Find the mean & variance of the distribution.

A). Given that,

$P(X=1) = 0.28336$

$P(X=2) = 0.0506$

ie, $n = 6$

$^6C_1\, p^{1}(1-p)^{6-1} = 0.28336.$

$^6C_1\, p(1-p)^5 = 0.28336$ —— (1)

$^6C_2\, p^2(1-p)^4 = 0.0506$ —— (2)

(1) ÷ (2) = 7

$$\frac{6p(1-p)^5}{15p^2(1-p)^4} = \frac{0.28336}{0.0506}$$

$$\frac{6(1-p)}{15p} = \frac{28}{5}$$

$30(1-p) = 420p$

$30 - 30p = 420p$

$30 = 450p$

$p = 1/15$

$\therefore n = 6$ & $p = 1/15$ & $1-p = 14/15$

Mean $= np$

$= 6/15 = 2/5$

Variance $= np(1-p)$

$= 2/5 \times 14/15$

$= 28/75$ //

Poisson Distribution

* The poisson distribution serves as a model for counts which do not have a natural upper bound.

Definition.

A discrete r.v X is defined to have poisson distribution if the ~~pdf~~ pmf of X is given by,

$$f(x) = \frac{e^{-\lambda}\lambda^{x}}{x!}, \; x = 0,1,2,\cdots, \; \lambda>0$$

$$= 0 \quad , \text{ otherwise.}$$

where λ is called the parameter of the poisson distribution.

Note:

$$\sum_{x=0}^{\infty} f(x;\lambda) = \sum_{x=0}^{\infty} \frac{e^{-\lambda}\lambda^{x}}{x!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^{x}}{x!}$$

$$= e^{-\lambda} \Big[\underbrace{\frac{\lambda^0}{0!} + \frac{\lambda^1}{1!} + \frac{\lambda^2}{2!} + \cdots}_{\text{Maclaurin's series for } e^{\lambda}} \Big]$$

$$= e^{-\lambda} e^{\lambda}$$

$$= 1$$

Mean of poisson distribution:

$$E(X) = \sum_{x=0}^{\infty} x \cdot f(x)$$

$$= \sum_{x=0}^{\infty} x \cdot \frac{e^{-\lambda}\lambda^{x}}{x!}$$

$$= \sum_{x=1}^{\infty} \frac{x e^{-\lambda}\lambda^{x}}{x!}$$

$$= \sum_{x=1}^{\infty} x \frac{e^{-\lambda}\lambda \cdot \lambda^{x-1}}{x(x-1)!}$$

$$= \lambda e^{-\lambda} \sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!}$$

$$= \lambda e^{-\lambda} e^{\lambda}$$

$$= \underline{\underline{\lambda}}$$

Mean $= \lambda$

Variance of poisson distribution.

Let x be a rv following poisson distribution with paremeter λ.

we've variance, $\sigma^2 = \sum_{i=1}^{n} (x_i - \mu)^2 p_i$

$$= E(X^2) - (E(X))^2$$

$$E(X(X-1)) = \sum_{i=0}^{\infty} x(x-1)\, p(x)$$

$$= \sum_{x=0}^{\infty} x(x-1) \frac{e^{-\lambda} \lambda^{x}}{x!}$$

$$= \sum_{x=2}^{\infty} x(x-1) \frac{e^{-\lambda} \lambda^{x-2} \cdot \lambda^{2}}{x(x-1)(x-2)!}$$

$$= e^{-\lambda} \lambda^{2} \sum_{x=2}^{\infty} \frac{\lambda^{x-2}}{(x-2)!}$$

$$= e^{-\lambda} \lambda^{2} \left[\frac{\lambda^{0}}{0!} + \frac{\lambda^{1}}{1!} + \frac{\lambda^{2}}{2!} + \cdots \right]$$

$$= e^{-\lambda} \lambda^{2} \cdot e^{\lambda}$$

$$= \underline{\underline{\lambda^{2}}}$$

$$E(X(X-1)) = \lambda^{2}$$

$$E(X^{2}) - E(X) = \lambda^{2}$$

$$E(X^{2}) = \lambda^{2} + E(X)$$

$$E(X^{2}) = \lambda^{2} + \lambda$$

$$\text{Variance} = E(X^{2}) - (E(X))^{2}$$

$$= \lambda^{2} + \lambda^{2} - \lambda^{2}$$

$$= \underline{\underline{\lambda}}$$

Variance $= \lambda$

The poisson Approximation to the Binomial Distribution.

The poisson distribution is obtained as an approximation to the binomial distribution under the conditions.

(1) n is very large $(n \to \infty)$

(2) p is very small $(p \to 0)$

(3) $np = \lambda$, a finite quantity.

proof:

For binomial distribution,

$$f(x) = {}^nC_x \, p^x (1-p)^{n-x}, \quad x = 0,1,2,\dots n.$$

$$= \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$$

$$= \frac{n(n-1)(n-2)\cdots(n-x)!}{x!\,(n-x)!} p^x (1-p)^{n-x}$$

$$= \frac{n(n-1)(n-2)\cdots(n-(x-1))}{x!} p^x (1-p)^{n-x}$$

$$= n(\underbrace{n \cdot n \cdots n}_{x-1 \text{ times}}) \frac{(1-1/n)(1-2/n)\cdots\left(1-\frac{(x-1)}{n}\right)}{x!} p^x (1-p)^{n-x}$$

$$= n \cdot n^{x-1} \frac{(1-1/n)(1-2/n)\cdots\left(1-\frac{(x-1)}{n}\right)}{x!} p^x (1-p)^{n-x}$$

~~Now~~, Now we've $\lambda = np$

~~then~~ $p = \lambda/n$

substitute $p = \lambda/n$ on $f(x)$.

$$f(x) = \frac{n^x \cdot (1-1/n)(1-2/n)\cdots(1-\frac{x-1}{n})}{x!} \frac{\lambda^x}{n^x}(1-\lambda/n)^{n-x}$$

$$= \frac{(1-1/n)(1-2/n)\cdots(1-\frac{x-1}{n})}{x!} \lambda^x (1-\lambda/n)^{n-x}$$

As $n \to \infty$ $1/n, 2/n, \cdots \frac{x-1}{n} \to 0$

$\therefore$

As $n \to \infty$,

$$\cancel{f(x) = 1 \cdot 1 \cdots 1 \, \frac{\lambda^x (1-0)^{n-x}}{x!}}$$

$$\cancel{= \lambda^x}$$

$$\cancel{f(x) = 1 \cdot \frac{\lambda^x (1-\frac{x-1}{n})^{n-x}}{x!}}$$

$$\cancel{= \frac{\lambda^x (1-\frac{x-1}{n})^n}{x!(1-\frac{x-1}{n})^x}}$$

$$\cancel{f(x) = \frac{\lambda^x (1-\frac{x-1}{n})^n}{x!(1-\frac{x-1}{n})^x}}$$

~~As $n \to \infty$, $\lim_{n\to\infty} 1-\lambda$~~

As $n \to \infty$.

$$f(x) = \frac{\lambda^x \left(1 - \lambda/n\right)^{n-x}}{x!}$$

$$\lim_{n \to \infty} \left(1 - \lambda/n\right)^n = e^{-\lambda}$$

$$= \frac{\lambda^x \left(1 - \lambda/n\right)^n}{x! \left(1 - \lambda/n\right)^x}$$

$$= \frac{\lambda^x e^{-\lambda}}{x!}$$

$$f(x) = \frac{\lambda^x e^{-\lambda}}{x!}, \quad x = 0, 1, 2, \ldots \infty$$

This is the pdf of poisson distribution. ie, $X \to P(\lambda)$. This shows that binomial distribution tends to poisson distribution.

Note:

* To find the probability of a success during a time interval of length T, we divide the interval into n equal parts of length Δt, so that $T = n \cdot \Delta T \Rightarrow n = T/\Delta T$.

The probability of a success during a very small interval of time Δt is given by $p = \alpha \cdot \Delta t$, where α is the average (mean) number of successes per unit time.

when $n \to \infty$ the probability of x successes during the time interval T is given by the corresponding poisson probability with the parameter

$$\lambda = n \cdot p = T/_{\Delta t} \cdot \alpha \Delta t = \alpha T$$

$$\lambda = \alpha T$$

2. If a bank receives on the avg $\alpha = 6$ bad checks per day, what are the probabilities that it will receive

always check the unit

a) 4 bad checks on any given day?

b) 10 bad checks over any 2 consecutive days?

A) Given that,

$\alpha = 6$. per day

a). $x = 4$, $T = 1$

$\lambda = \alpha T = 1 \cdot 6 = 6$.

$$P(x : \lambda) = \frac{e^{-\lambda} \lambda^{x}}{x!}$$

$$f(4 : 6) = \frac{e^{-6} \, 6^{4}}{4!} = 0.134$$

b) $x = 10$, $T = 2$, $\lambda = 2 \times 6 = 12$

$$F(10:12) = \frac{e^{-12}(12)^{10}}{10!}$$

$$= \underline{\underline{0.105}}$$

? In the inspection of tin plate produced by a continuous electrolytic process, 0.2 imperfection is spotted per minute, on average. Find the probabilities of spotting

a) one imperfection in 3 minutes;

b) at least two imperfection in 5 minutes.

c) at most one imperfection in 15 minutes.

A). Given,

$\alpha = 0.2$ / minutes

a) $x=1$, $T=3$, $\lambda = \alpha T = 0.2 \times 3 = 0.6$

$$f(x:\lambda) = \frac{e^{-\lambda}\lambda^{x}}{x!}$$

$$f(1:0.6) = \frac{e^{-0.6}(0.6)^1}{1!}$$

$$= \underline{\underline{0.3293}}$$

b) $x = 2, 3, \cdots \infty$, $T=5$, $\lambda = 5.02 = \frac{1}{\cancel{8}}$//

by
ection
he

$$f(x \geq 2 : \lambda) = 1 - f(x < 2 ; \lambda)$$

$$= 1 - f(0 : \lambda) + f(1 : \lambda)$$

$$= 1 - f(0:1) + f(1:1)$$

$$= 1 - \frac{e^{-1} (1)^0}{0!} + \frac{e^{-1} (1)^1}{1!}$$

$$= 1 - e^{-1} + e^{-1}$$

$$= 1 - 2e^{-1}$$

$$= \underline{\underline{0.2642}}$$

c) $x = 0,1$, $T = 15$, $\lambda = 0.2 \times 15 = 3$.

$$F(x = 0,1 : \lambda) = f(0 : 3) + f(1 : 3)$$

$$= \frac{e^{-3} 3^0}{0!} + \frac{e^{-3} \cdot 3^1}{1!}$$

$$= e^{-3} + 3e^{-3}$$

$$= 4e^{-3}$$

$$= \underline{\underline{0.1991}}$$

2. If electricity power failures occur according to a poisson distribution with an avg of 3 failures every 20 weeks. calculate the probability that there will not be more than one failure during a particular week.

A). $\alpha = 3$ failure per 20 weeks,

~~3 f~~ avg no.of failure per week is $\alpha = 3/20 = 0.15$

$\alpha = 0.15$

we have to find the probability of not more than one failure during a particular week.

ie,

$P(X \leq 1 : \lambda) = ?$ $\quad \lambda = 0.15 \times 1 = 0.15$

$P(X \leq 1 ; \lambda) = f(0 : 0.15) + f(1 : 0.15)$

$= \dfrac{e^{-0.15} (0.15)^0}{0!} + \dfrac{e^{-0.15} (0.15)}{1!}$

$= e^{-0.15} + 0.15 \times e^{-0.15}$

$= 1.15 \times e^{-0.15}$

$= \underline{\underline{0.9898}}$

2. There are 250 typographical errors in a book of 1000 pages. The no.of errors per page is supposed to follow a poisson distribution. what is the probability a randomly selected page will have more than 2 errors?

A) The avg no. of errors per one page $= \dfrac{\cancel{100}\ 250}{1000} = 0.25$

$\alpha = 0.25$ $\lambda = 0.25 \times 1 = 0.25$

$p(x>2:\lambda) = ?$

$p(x>2:\lambda) = 1 - p(x \leq 2:\lambda)$

$= 1 - p(0:0.25) + p(1:0.25) + p(2:0.25)$

$= 1 - \frac{e^{-0.25}(0.25)^0}{0!} + \frac{e^{-0.25} \cdot (0.25)^1}{1!} + \frac{e^{-0.25}(0.25)^2}{2!}$

$= 1 - e^{-0.25} + e^{-0.25}\, 0.25 + 0.25^2\, e^{-0.25}$

$= 1 - 1.3125\, e^{-0.25}$

$= \underline{0.0021}$

? A statistician records the no. of cars that approach an intersection. He finds that an avg of 1.6 cars approach the intersection every minute.
Assuming the no. of cars that approach this intersection follows a poisson distribution, what is the probability that 3 or more cars will approach the intersection within a minute?

A $\alpha = 1.6$ /minute. $\lambda = 1.6 \times 1 = 1.6$

$x = 3, 4 \ldots \infty$.

$P(x \geq 3 : 1.6) = 1 - P(x < 3 : 1.6)$

$=$

$= 1 - [P(0:1.6) + P(1:1.6) + P(2:1.6)]$

$= 1 - \left[\frac{e^{-1.6}(1.6)^0}{0!} + \frac{e^{-1.6}(1.6)^1}{1!} + \frac{e^{-1.6}(1.6)^2}{2!}\right]$

$= 1 - \left[e^{-1.6} + 1.6e^{-1.6} + \frac{e^{-1.6}(1.6)^2}{2}\right]$

$= \underline{\underline{0.2166}}$

2. If x is a poisson variable such that $P(x=2) = P(x=3)$ then find $P(x=4)$.

A) Given $x \sim \text{Poisson}(\lambda)$

$P(x) = f(x) = \frac{\lambda^x e^{-\lambda}}{x!}$

$P(x=2) = P(x=3) \Rightarrow \frac{e^{-\lambda}\lambda^2}{2!} = \frac{e^{-\lambda}\lambda^3}{3!}$

$\frac{1}{2} = \lambda/6$

$\underline{\underline{\lambda = 3}}$

we've to find $P(x=4)$,

$P(x=4) = \cancel{e^{-3}\lambda} \quad \frac{e^{-\lambda}\lambda^x}{x!}$

$= \cancel{e^{-3}4^3}$

$= \frac{e^{-3}3^4}{4!} = \underline{\underline{0.1680}}$

Uniform Distribution:

Uniform distribution is a probability distribution that asserts that the outcomes for a discrete set of data have the same probability.

Example:

(1). A deck of cards has within it uniform distributions, bcoz the likelihood of drawing a heart, a club, a diamond, or a spade is equally likely.

(2) A coin also has a uniform distribution bcz the probability of getting either heads or tails in a coin toss is the same.

Definition

Let X be a discrete r.v following uniform distribution with parameter n & possible values of $x = 1, 2, 3, \ldots, n$. Then $P(X=x)$ is given by,

$$P(x) = \begin{cases} 1/n, & x = 1, 2, \ldots n \\ 0, & \text{o.w} \end{cases}$$

where n is the no. of values in the range set.

Then we say $X \sim U(n)$.

Mean of uniform distribution.

Let $X \sim U(n)$.

Mean of X is given by. $\mu = \frac{n+1}{2}$

proof,

$$\text{Mean} = E(X) = \sum_{x=1}^{\infty} x \cdot P(x)$$

$$= \sum_{x=1}^{\infty} x \cdot 1/n$$

$$= 1/n \, [1 + 2 + 3 + \cdots]$$

$$= \frac{1}{n} \frac{n(n+1)}{2}$$

$$= \frac{n+1}{2}$$

Variance of uniform distribution.

Let $X \sim U(n)$,

Then variance of x is given by $Var(x) = \frac{n^2-1}{12}$.

proof,

$$V(X) = E(X^2) - [E(X)]^2$$

$$E(X^2) = \sum_{x=1}^{\infty} x^2 \cdot 1/n$$

$$= 1/n \, [1 + 2^2 + 3^2 + \cdots]$$

$$= \frac{1}{n} \frac{n(n+1)(2n+1)}{6}$$

$$V(X) = \frac{n(n+1)(2n+1)}{6n} - \left(\frac{n+1}{2}\right)^2$$

$$= \frac{n+1}{2}\left[\frac{2n+1}{3} - \frac{n+1}{2}\right]$$

$$= \frac{n+1}{2}\left[\frac{4n+2-3n-3}{6}\right]$$

$$= \left(\frac{n+1}{2}\right)\frac{(n-1)}{6}$$

$$= \frac{n^2-1}{12}$$

Geometric distribution.

(1) Assume Bernoulli trials, ie, there are 2 possible outcomes.

(2) The trials are independent

(3). Let p be the probability of success which remains the same from trial to trial.

Let x denote the no. of trials until the first success. Then, the pmf of x is,

$$g(x;p) = p(1-p)^{x-1}, \quad x = 1,2,3,\dots$$

Thus $g(x;p)$ gives the probability of getting the 1st success on the x^{th} trial.

The probability distribution is called the geometric distribution.

Mean of geometric distribution.

Mean of geometric distribution is given by,

$\mu = 1/p$

$$\text{Mean} = E(X) = \Sigma x f(x)$$
$$= \sum_{x=1}^{\infty} x \cdot p(1-p)^{x-1}$$
$$= p \sum_{x=1}^{\infty} x(1-p)^{x-1}$$
$$= p[1 + 2(1-p) + 3(1-p)^2 + \dots]$$
$$1 + 2x + 3x^2 + \dots = (1-x)^{-2}$$
$$= p(1-(1-p))^{-2}$$
$$= \frac{p}{(1 \mp 1 + p)^2}$$
$$= p/p^2 = 1/p$$

Variance:

Variance of geometric distribution is given by $\text{var}(X) = \frac{1-p}{p^2}$.

Proof,

$$V(X) = E(X^2) - [E(X)]^2$$

First we find $E(X(X+1))$

$$E(X(X-1)) = \sum_{x=1}^{\infty} x(x-1)p(1-p)^{x-1}$$

$$= p\sum_{x=1}^{\infty} x(x-1)(1-p)^{x-1}$$

$$= p[0 + 2(1-p) + 6(1-p)^2 + 12(1-p)^3 + \cdots]$$

$$= p(2(1-p))[1 + 3(1-p) + 6(1-p)^2 + \cdots]$$

$1 + 3x + 6x^2 \cdots = (1-x)^{-3}$

$$= \frac{2p(1-p)}{(1-(1-p))^3}$$

$$= \frac{2p(1-p)}{p^3}$$

$$= \frac{2(1-p)}{p^2}$$

$$E(X(X-1)) = \frac{2(1-p)}{p^2}$$

$$E(X^2) - E(X) = \frac{2(1-p)}{p^2}$$

$$E(X^2) = \frac{2(1-p)}{p^2} + 1/p$$

$$= \frac{2(1-p)}{p^2} + \frac{p}{p^2}$$

$$= \frac{2(1-p)+p}{p^2}$$

$$V(X) = E(X^2) - (E(X))^2$$

$$= \frac{2(1-p)+p}{p^2} - \frac{1}{p^2}$$

$$= \frac{2-2p+p-1}{p^2}$$

$$= \frac{1-p}{p^2}$$

? Suppose that a trainee soldier shoots a target in an independent fashion. If the probability that the target is hit in any shot is 0.7, then what is the avg no. of shots needed to hit the target?

A). avg no $= 1/p = \frac{1}{0.7} = 1.4286$

? Products produced by a machine has a 3% defective rate.

(1) what is the probability that the 1st defective occurs in the fifth item inspected?

(2) what is the probability that the 1st defective does not occurs in the 1st five item inspected?

A) Given,

$P = 3\% = 0.03$

a) $x = 5$.

$g(x:p) = p(1-p)^{x-1}$

$g(5:0.03) = (0.03)(1-0.03)^4$

$= 0.02656$

(b) ~~x≠1,2,3,4~~, 5

$x \neq 1, 2, 3, 4, 5.$

$g(x>5; p) = ?$

$g(x>5) = 1 - g(x \leq 5)$

$= 1 - (0.03 + 0.03 \times 0.97 + 0.03 \times (0.97)^2 +$
$0.03 \times (0.97)^3 + 0.03 \times (0.97)^4)$

$= 0.8587$

OR

$(1 - 0.03)^5 = (0.97)^5$

conditional probability

Let A & B any 2 events in the sample space S with $P(B) \neq 0$. Then the conditional probability of A given that B has already occurred is denoted by, $P(A/B)$ & is defind by,

$$P(A \setminus B) = \frac{P(A \cap B)}{P(B)}, \quad P(B) \neq 0$$

2. Two dies are thrown simultaneously & the sum of the numbers obtained was found to be 7. what is the probability that

the number 3 has appeared atleast once?

A) Sample space,

$$S = \begin{Bmatrix} (1,1), & \dots & (1,6) \\ (2,1). & \dots & (2,6) \\ \vdots & & \\ (6,1) & \dots & (6,6) \end{Bmatrix} (6 \times 6 = 36)$$

B = sum of ~~probo~~ the numbers is 7

A = number 3 has appeared atleast once.

$$P(A \setminus B) = \frac{P(A \cap B)}{P(B)}$$

$B = \{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\}$

$A = \{(1,3), (2,3), (3,1), (3,2), (3,3), (3,4), (3,5), (3,6),$
$(4,3), (5,3), (6,3)\}$

$P(A \cap B) = 2/36$

$P(B) = 6/36$

$$P(A \setminus B) = \frac{2/36}{6/36} = \underline{\underline{1/3}}$$

required probability = 1/3

Q. X is a r.v with pmf $f(x) = \frac{x}{10}$, $x = 1, 2, 3, 4$

Find

(i) $P(X \leq 2)$

(ii) $P(1/2 < X < 5/2 \mid X > 1)$

(iii) the cumulative distribution fn (cdf) of X

A) Given. $f(x) = \frac{x}{10}$, $x = 1, 2, 3, 4$

(i) $P(X \leq 2) = \frac{1}{10} + 2/10 = \underline{\underline{3/10}}$

(2). $P(1/2 < X < 5/2 \mid x > 1)$

$1/2 < X < 5/2 \Rightarrow x = 2$

~~$P(X=2)$~~

$$P(1/2 < X < 5/2 \mid x > 1) = P(X = 2 \mid x > 1)$$

$$= \frac{P(X = 2 \;\&\; x > 1)}{P(X > 1)}$$

$$= \frac{P(X=2)}{P(X>1)}$$

$$= \frac{P(X=2)}{1 - P(X \leq 1)}$$

$$= \frac{2/10}{1 - 1/10} = \frac{2/10}{9/10} = \underline{\underline{2/9}}$$

(iii) Let $F(x) = P(X \leq x)$ denote the cdf

$\therefore F(1) = P(X \leq 1) = 1/10$

$F(2) = P(X \leq 2) = \frac{1}{10} + \frac{2}{10} = 3/10$

$F(3) = P(X \leq 3) = \frac{1}{10} + \frac{2}{10} + \frac{3}{10} = 6/10$

$F(4) = P(X \leq 4) = \frac{1}{10} + \frac{2}{10} + 3/10 + 4/10 = 10/10 = 1$

$\therefore$ we've

$X = x$	1	2	3	4
$f(x) = P(X = x)$	1/10	2/10	3/10	4/10
$F(x) = P(X \leq x)$	1/10	3/10	6/10	1

cdf can be written as,

$$F(x) = \begin{cases} 0 & ; x < 1 \\ \frac{1}{10} & ; 1 \leq x < 2 \\ 3/10 & ; 2 \leq x < 3 \\ 6/10 & ; 3 \leq x < 4 \\ 1 & ; x \geq 4 \end{cases}$$

Madale 2

A

2)